# On measures driven by Markov chains

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# ON MEASURES DRIVEN BY MARKOV CHAINS

YANICK HEURTEAUX AND ANDRZEJ STOS

Abstract. We study measures on $[0,1]$ which are driven by a finite Markov chain and which generalize the famous Bernoulli products. We propose a hands-on approach to determine the structure function $\tau$ and to prove that the multifractal formalism is satisfied. Formulas for the dimension of the measures and for the Hausdorff dimension of their supports are also provided.

## 1. Introduction

Multifractal measures on $\mathbb{R}^d$ are measures m for which the level sets

$$E_\alpha = \left\{ x \in \mathbb{R}^d \ ; \lim_{r\to 0} \frac{\log \mathrm{m}(B(x,r))}{\log r} = \alpha \right\}$$

are non-trivial for at most two values of the real $\alpha$. In practice, it is impossible to completely describe the sets $E_\alpha$, but we can try to calculate their Hausdorff dimensions. To this end, Frisch and Parisi ([7]) were the first to use the Legendre transform of a structure function $\tau$. A mathematically rigorous approach was given by Brown, Michon and Peyrière in [2] and by Olsen in [9]. There are many situations in which the Legendre transform formula, now called multifractal formalism, is satisfied. For a comprehensive account see see e.g. [3], [5], [6], [11] or [8].

A fundamental model for multifractal measures is given by so called Bernoulli products, see for example Chapter 10 of [4]. Roughly speaking, if $I_{\varepsilon_1\cdots\varepsilon_n}$ are the $\ell$-adic intervals of the $n^{th}$ generation and $(X_n)$ is an i.i.d. sequence of random variables, then Bernoulli product m can be defined by

$$\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = P[X_1 = \varepsilon_1, \cdots, X_n = \varepsilon_n].$$

The purpose of this paper is to provide an explicit analysis of a natural generalization of this model. Instead of an i.i.d. sequence, we consider an irreducible homogeneous Markov chain. Consequently, the measure m satisfies the recurrence relation

$$\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_{n+1}}) = p_{\varepsilon_n\varepsilon_{n+1}}\, \mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}),$$

where $P = (p_{ij})$ is the transition matrix of $X_n$ (see the next section for full details).

In Section 3, we identify a formula for the structure function $\tau$ of such a measure and we compute its dimension. Section 4 contains a construction of auxiliary (Gibbs) measures. While it involves a nontrivial rescaling, we insist on the fact that our results don't require sophisticated tools but only some fundamental results of multifractal analysis and the use of Perron-Frobenius theorem. We are then able to prove that the multifractal formalism is satisfied and we give a formula for the Hausdorff dimension of the (closed) support of the measures. Finally, in Section 5

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we discuss ergodic properties and prove that for a given support $K$, the measure m with maximal dimension is essentially unique.

## 2. Preliminaries

Set $\mathcal{S} = \{0, 1, 2, \ell - 1\}$. Let $\mathcal{W}_n$ be set of words of length $n$ over the alphabet $\mathcal{S}$. A concatenation of two words $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$ and $\delta = \delta_1 \cdots \delta_k \in \mathcal{W}_k$ will be denoted by $\varepsilon\delta = \varepsilon_1 \cdots \varepsilon_n \delta_1 \cdots \delta_k$. Let $\mathcal{F}_0 = \{[0, 1)\}$ and for $n \geq 1$, $\mathcal{F}_n$ be the set of $\ell$-adic intervals of order $n$, that is the family of intervals of the form

$$I_\varepsilon = I_{\varepsilon_1 \cdots \varepsilon_n} = \left[ \sum_{i=1}^{n} \frac{\varepsilon_i}{\ell^n}, \sum_{i=1}^{n} \frac{\varepsilon_i}{\ell^n} + \frac{1}{\ell^n} \right)$$

where $\varepsilon = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{W}_n$. If $I = I_\varepsilon \in \mathcal{F}_n$ and $J = I_\delta \in \mathcal{F}_k$, we will write $IJ = I_{\varepsilon\delta}$. Note that $IJ \subset I$. Finally, we will denote by $I_n(x)$ the unique interval $I \in \mathcal{F}_n$ such that $x \in I$ and $|I|$ the length of the interval $I$.

Consider a discrete Markov chain $X = (X_k)_{k \geq 1}$ on $\mathcal{S}$ with an initial distribution $p_i = P(X_1 = i)$, $i \in \mathcal{S}$, and the transition probabilities $P = (p_{ij})_{i,j=0}^{\ell-1}$ where $p_{ij} = P(X_{n+1} = j \,|\, X_n = i)$. In order to exclude degenerated cases, we will suppose that $p_{ij} \neq 1$ for any $(i, j) \in \mathcal{S} \times \mathcal{S}$. Note that for the sake of coherence with the definition of $\mathcal{F}_n$, the entries of matrices and vectors will be indexed by numbers starting from 0.

If necessary, we will assume that $X$ is irreducible, that is for all $i, j \in S$ there exists $k \geq 2$ such that $P(X_k = j | X_1 = i) > 0$.

For a matrix $P = (p_{ij})_{i,j \in \mathcal{S}}$ we denote by $P^n$ the usual matrix power. This should not be confused with $P_q$, $q \in \mathbb{R}$, which stands for the matrix whose entries are given by $(p_{ij}^q)_{i,j \in \mathcal{S}}$. In this context, by convention we will set $0^q = 0$ for any $q \in \mathbb{R}$.

The Hausdorff dimension (box dimension, respectively) of a set $E$ will be denoted by $\dim_H E$ ($\dim_B E$). Recall the definitions of the lower and upper dimension of a probability measure $\mu$ on $\mathbb{R}^d$:

$$\dim_* \mu = \inf\{\dim_H(E) : \mu(E) > 0\} = \sup\{s > 0 : \mu \ll \mathcal{H}^s\}$$
$$\dim^* \mu = \inf\{\dim_H(E) : \mu(E) = 1\} = \inf\{s > 0 : \mu \perp \mathcal{H}^s\}$$

where $\mathcal{H}^s$ denotes the Hausdorff measure in dimension $s$. If these two quantities agree, the common value is called the dimension of $\mu$ and denoted by $\dim \mu$. In this case the measure is said to be *unidimensional*. For more details, we refer the reader to e.g. [8].

By $c$ or $C$ we will denote a generic positive constant whose exact value is not important and may change from line to line. For functions or expressions $f$ and $g$, depending on a variable $x$, say, we will write $f \asymp g$ if there exists a constant $c$ which does not depend on $x$ and such that $c^{-1} g(x) \leq f(x) \leq c g(x)$ for any admissible $x$.

## 3. The measures and the structure function $\tau$

Let $(X_n)_{n \geq 1}$ be a Markov chain on $\mathcal{S}$ with initial distribution $(p_i)_{i \in S}$ and transition matrix $P = (p_{ij})$. Define the measure m as follows:

$$\mathrm{m}(I_{\varepsilon_1 \cdots \varepsilon_n}) = P(X_1 = \varepsilon_1, \ldots, X_n = \varepsilon_n).$$

Then $\mathrm{m}(I_i) = p_i$, $i \in \mathcal{S}$, and by the Markov property,

$$\begin{aligned}\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) &= P(X_n = \varepsilon_n \mid X_{n-1} = \varepsilon_{n-1} \cdots X_1 = \varepsilon_1) P(X_{n-1} = \varepsilon_{n-1} \cdots X_1 = \varepsilon_1) \\ &= p_{\varepsilon_{n-1}\varepsilon_n}\, \mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_{n-1}}).\end{aligned}$$

Iterating, we get

$$\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = p_{\varepsilon_1} p_{\varepsilon_1\varepsilon_2} \cdots p_{\varepsilon_{n-1}\varepsilon_n}. \tag{3.1}$$

In other words, a finite trajectory of $X$ selects an interval and assigns it a mass equal to the probability of the trajectory. The measure is well defined since for any given $\ell$-adic interval $I$ there is exactly one path to it and its subintervals can be reached only through $I$.

Because of the additivity property $\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = \sum_{k=0}^{n-1} \mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_{n-1}k})$ and the property $\lim_{n\to+\infty} \mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = 0$, it is well known that formula (3.1) defines a probability Borel measure whose support is contained in $[0,1]$ (see for example [12]). Moreover the measure m is such that $\mathrm{m}(\{x\}) = 0$ for any point $x$.

The construction proposed here can be viewed as a generalization of a classical Bernoulli measure. Our goal is to give a hands-on approach to the multifractal analysis of such measures.

**Example 3.1.** (1) Natural measure on the triadic Cantor set. Let

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and set the initial distribution $(p_0, p_1, p_2) = (1/2, 0, 1/2)$. Then $\mathcal{C} = \operatorname{supp} \mathrm{m}$ is the ternary Cantor set and m is the normalized $(\log_3 2)$-Hausdorff measure on $\mathcal{C}$.

(2) Bernoulli measures. Let $p = (p_0, \cdots, p_{\ell-1})$ be a probability vector and suppose that $(X_i)$ are i.i.d. random variables with $P(X_1 = j) = p_j$. By independence $p_{ij} = p_j$ so that

$$P = \begin{pmatrix} p_0 & p_1 & \cdots & p_{\ell-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ p_0 & p_1 & \cdots & p_{\ell-1} \end{pmatrix}.$$

Hence, by (3.1) we get $\mathrm{m}(I_{\varepsilon_1\cdots\varepsilon_n}) = p_{\varepsilon_1} \cdots p_{\varepsilon_n}$, and the measure m is the classical Bernoulli product.

(3) Let $\ell = 2$, $p \neq 1/2$ and

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

The associated measure m was introduced by Tukia in [13]. It is a doubling measure with dimension $\dim(\mathrm{m}) = -(p\log_2 p + (1-p)\log_2(1-p)) < 1$. The associated repartition function $f(x) = \mathrm{m}([0,x])$ is a singular quasisymetric function.

(4) Random walk on $\mathbb{Z}_\ell$. Let $p_{ij} = 1/2$ if $|i-j| = 1$ or $(i,j) = (0, \ell-1)$ or $(i,j) = (\ell-1, 0)$. If, for example, $n = 3$,

$$P = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

As we will see later, the associated measure is monofractal with dimension $\frac{\log 2}{\log \ell}$.

Define as usual the structure function $\tau(q)$ by

$$\tau(q) = \limsup_{n\to\infty} \frac{1}{n\log\ell} \log\left(\sum_{I\in\mathcal{F}_n} \mathrm{m}(I)^q\right), \tag{3.2}$$

with the usual convention $0^q = 0$ for any $q \in \mathbb{R}$.

**Theorem 3.2.** *Let* m *be a probability measure driven by a Markov chain with transition matrix $P$. Suppose that the matrix $P$ is irreductible. For $q \in \mathbb{R}$, let $\lambda_q$ be the spectral radius of $P_q$. Then $\tau(q) = \log_\ell(\lambda_q)$ and the limit does exist in* (3.2).

*Proof.* Let $\mathcal{W}_{n,k}$ be the subset of $\mathcal{W}_n$ consisting of words that end with $k \in \mathcal{S}$. Set

$$s_{n,k} = \sum_{\varepsilon\in\mathcal{W}_{n,k}} \mathrm{m}(I_\varepsilon)^q$$

and let $S_n$ be the (line) vector $(s_{n,0}, \dots, s_{n,\ell-1})$. In particular, $\mathcal{W}_{1,k} = \{k\}$ and $S_1 = (p_0^q, \dots, p_{\ell-1}^q)$. We claim that

$$S_n P_q = S_{n+1}, \qquad n \geq 1. \tag{3.3}$$

Indeed, the $j^{\text{th}}$ coordinate of $S_n P_q$ is given by

$$\sum_{k\in\mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k\in\mathcal{S}} \sum_{\varepsilon\in\mathcal{W}_{n,k}} p_{kj}^q \,\mathrm{m}(I_\varepsilon)^q$$

Using the Markov property, we have $p_{kj}\,\mathrm{m}(I_\varepsilon) = \mathrm{m}(I_{\varepsilon j})$ when $\varepsilon \in \mathcal{W}_{n,k}$. It follows that

$$\sum_{k\in\mathcal{S}} s_{n,k} p_{kj}^q = \sum_{k\in\mathcal{S}} \sum_{\varepsilon\in\mathcal{W}_{n,k}} \mathrm{m}(I_{\varepsilon j})^q = \sum_{\varepsilon\in\mathcal{W}_{n+1,j}} \mathrm{m}(I_\varepsilon)^q = s_{n+1,j}$$

which is the $j^{\text{th}}$ coordinate of $S_{n+1}$. So that (3.3) follows. Iterating, we obtain

$$S_n = S_1 \left(P_q\right)^{n-1}. \tag{3.4}$$

Now, observe that

$$\sum_{\varepsilon\in\mathcal{W}_n} \mathrm{m}(I_\varepsilon)^q = \sum_{k\in\mathcal{S}} s_{n,k} = \|S_n\|_1 = \|S_1 P_q^{n-1}\|_1$$

where $\| \ \|_1$ is the $\ell^1$ norm in $\mathbb{R}^\ell$. It follows that

$$\begin{aligned}\tau(q) &= \limsup_{n\to\infty} \frac{1}{n} \log_\ell \left( \sum_{\varepsilon\in W_n} \mathrm{m}(I_\varepsilon)^q \right) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log_\ell \|S_1 P_q^{n-1}\|_1 \\ &= \limsup_{n\to\infty} \frac{1}{n} \log_\ell \|S_1 P_q^{n}\|_1\end{aligned}$$

Let us now introduce the following notation. If $a = (a_0, \cdots, a_{\ell-1})$ and $b = (b_0, \cdots, b_{\ell-1})$ are two vectors in $\mathbb{R}^\ell$, we will write $a \prec b$ when $a_i \le b_i$ for any value of $i$. Observe in particular that that $S_1 \succ 0$ and is not identically equal to 0. Let $\lambda_q$ be the spectral radius of the matrix $P_q$, which is also the spectral radius of the transposed matrix $P_q^t$. The matrix $P_q$ being positive and irreductible, Perron-Frobenius Theorem ensures the existence of an eigenvector $\nu_q$ with strictly positive entries, satisfying $\nu_q P_q = \lambda_q \nu_q$. Therefore, there exists a constant $C > 0$ such that $S_1 \prec C\nu_q$. It follows that

$$\|S_1 P_q^n\|_1 \le C\|\nu_q P_q^n\|_1 = C\|\nu_q\|_1 \lambda_q^n.$$

On the other hand, using the irreductibility property of the matrix $P_q$, we can find an integer $k$ such that the matrix $I + P_q + \cdots + P_q^k$ has strictly positive entries. It follows that the line vector $S_1 + S_1P_q + \cdots + S_1P_q^k$ has strictly positive entries and we can find a constant $C > 0$ such that

$$\nu_q \prec C \left( S_1 + S_1P_q + \cdots + S_1P_q^k \right).$$

So

$$\begin{aligned}\|\nu_q P_q^n\|_1 &\le C\|S_1P_q^n + S_1P_q^{n+1} + \cdots + S_1P_q^{n+k}\|_1 \\ &= C\| \left(S_1P_q^n\right)\left(I + P_q + \cdots P_q^k\right) \|_1 \\ &\le C'\|S_1P_q^n\|_1.\end{aligned}$$

Finally, $\|S_1P_q^n\|_1 \asymp \lambda_q^n$.Taking the logarithm, we can conclude that $\tau(q) = \log_\ell(\lambda_q)$ and that the limit exists.

□

**Corollary 3.3.** *The function $\tau$ is analytic on $\mathbb{R}$.*

*Proof.* This can be seen as a consequence of the Kato-Rellich theorem (see for example [10]). But in this finite dimensional context, there is an elementary proof. Let $F(q, x) = \det(P_q - xI)$ be the characteristic polynomial of $P_q$ and let $q_0 \in \mathbb{R}$. Observing that $F(q_0, \lambda_{q_0}) = 0$ and $\frac{\partial F}{\partial x}(q_0, \lambda_{q_0}) \ne 0$ (the eigenvalue $\lambda_{q_0}$ is simple), the map $q \mapsto \lambda_q$ is given arround $q_0$ by the implicit functions theorem. Moreover, $F$ being analytic in $q$ and $x$, it is well known that the implicit function is analytic. □

The existence of $\tau'(1)$ ensures that the measure m is unidimensional (see e.g. [8], Theorem 3.1).

**Corollary 3.4.** *The measure* m *is unidimensional with dimension* $\dim(\mathrm{m}) = -\tau'(1)$.

Let us now describe some examples. Let $h_\ell$ be the usual entropy function

$$h_\ell(p) = -\sum_{i=0}^{\ell-1} p_i \log_\ell p_i, \qquad p = (p_0, \ldots, p_{\ell-1}) \text{ with } \sum_{i=0}^{\ell-1} p_i = 1.$$

In particular, set $h(x) = h_2(x, 1-x) = -(x\log_2(x) + (1-x)\log_2(1-x))$.

**Example 3.5.** If m is the Bernoulli measure from Example 3.1 (2), then by Theorem 3.2 we get the well known formula for $\tau$

$$\tau(q) = \log_\ell\left(p_0^q + \cdots + p_{\ell-1}^q\right).$$

Furthermore, the dimension of the measure m is $\dim(\mathrm{m}) = -\tau'(1) = h_\ell(p)$.

**Example 3.6.** Actually, if $\ell = 2$, we can obtain an explicit formula for any given Markov chain. Suppose that $a, b \in (0,1)$ and

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

Then

$$\dim(\mathrm{m}) = \frac{b}{a+b}h(a) + \frac{a}{a+b}h(b). \tag{3.5}$$

Indeed, by Theorem 3.2 we get

$$\tau(q) = -1 + \log_2\left((1-a)^q + (1-b)^q + \sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q}\right).$$

Note that if $q = 1$, then $\sqrt{((1-a)^q - (1-b)^q)^2 + 4a^q b^q}$ simplifies to $a + b$. Thus, we obtain

$$\tau'(1) = \frac{1}{2\log 2}\Bigg((1-a)\log(1-a) + (1-b)\log(1-b) + \\ + \frac{(b-a)((1-a)\log(1-a) - (1-b)\log(1-b)) + 2ab\log ab}{a+b}\Bigg)$$

Rearranging, we get (3.5). Note that the coefficients in (3.5) come from the stationary distribution of $X$:

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right).$$

This observation will be generalized in Theorem 3.8 below.

**Example 3.7.** Let $(a_0, \cdots, a_{\ell-1})$ be a probability vector (with possibly some entries that are equal to 0) and $P$ be an $\ell \times \ell$ irreductible stochastic matrix with entries $a_0, \ldots, a_{\ell-1}$ in every row, but in an arbitrary order. Then $\tau(q) = \log_\ell\left(a_0^q + \ldots + a_{\ell-1}^q\right)$ and $\dim(\mathrm{m}) = h_\ell(a_0, \cdots, a_{\ell-1})$. In particular, if $\kappa$ is the number of $a_i$'s that are not equal to 0 and if each nonzero $a_i$ is equal to $1/\kappa$, we get $\dim(\mathrm{m}) = \frac{\log\kappa}{\log\ell}$ which is the maximal possible value and is also the dimension of the support of the measure m. Such a remark will be generalized below.

*Proof.* Set $A_q = a_0^q + \cdots + a_{\ell-1}^q$. We have

$$\begin{aligned}\sum_{\varepsilon\in\mathcal{W}_{n+1}} \mathrm{m}(I_\varepsilon)^q &= \sum_{k\in\mathcal{S}}\sum_{\varepsilon\in\mathcal{W}_{n,k}}\sum_{j\in\mathcal{S}} p_{kj}^q\,\mathrm{m}(I_\varepsilon)^q \\ &= A_q\sum_{k\in\mathcal{S}}\sum_{\varepsilon\in\mathcal{W}_{n,k}}\mathrm{m}(I_\varepsilon)^q \\ &= A_q\sum_{\varepsilon\in\mathcal{W}_n}\mathrm{m}(I_\varepsilon)^q.\end{aligned}$$

Iterating, we get

$$\sum_{\varepsilon\in\mathcal{W}_n} \mathrm{m}(I_\varepsilon)^q = A_q^{n-1} \sum_{\varepsilon\in\mathcal{W}_1} \mathrm{m}(I_\varepsilon)^q.$$

Consequently,

$$\tau(q) = \log_\ell\left(a_0^q + \ldots + a_{\ell-1}^q\right).$$

It follows that $\tau'(1) = -h_\ell\,(a_0, \ldots, a_{\ell-1})$.

If there are only $\kappa$ nonzero values in the probability vector $(a_0, \cdots, a_{\ell-1})$, say e.g. $a_0, \cdots, a_{\kappa-1}$, the formula turns to $\dim \mathrm{m} = -\sum_{j=0}^{\kappa-1} a_j \log_\ell a_j$ which is maximal when $a_j = 1/\kappa$ for any $j$. □

Let us finish this part with a general formula for the dimension of the measure m. That is the purpose of the following theorem.

**Theorem 3.8.** *Denote by $L_k$ the $k^{th}$ line of the matrix $P$. Let $H = (h_\ell(L_0), \ldots, h_\ell(L_{\ell-1}))$ be the vector of entropies of the lines of $P$. Then*

$$\dim(\mathrm{m}) = <\pi\,|\,H>,$$

*where $\pi$ is the stationary distribution of the Markov chain $X$ and $<\ |\ >$ is the canonical scalar product.*

*Proof.* Let $T_n = \sum_{I\in\mathcal{F}_n} \mathrm{m}(I)\log_\ell \mathrm{m}(I)$, and $H_n = \frac{-1}{n}T_n$ the entropy related to the partition $\mathcal{F}_n$. We know that $\tau'(1)$ exists. It follows that $-\tau'(1) = \lim_{n\to+\infty} H_n$ (see for example [8], Theorem 3.1). Further,

$$\begin{aligned} T_n &= \sum_{k\in\mathcal{S}} \sum_{\varepsilon\in\mathcal{W}_{n-1,k}} \sum_{j\in\mathcal{S}} \mathrm{m}(I_\varepsilon)p_{kj}\log_\ell(\mathrm{m}(I_\varepsilon)p_{kj}) \\ &= -\sum_{k\in\mathcal{S}} h_\ell(L_k) \sum_{\varepsilon\in\mathcal{W}_{n-1,k}} \mathrm{m}(I_\varepsilon) + \sum_{k\in\mathcal{S}} \sum_{\varepsilon\in\mathcal{W}_{n-1,k}} \mathrm{m}(I_\varepsilon)\log_\ell(\mathrm{m}(I_\varepsilon)) \end{aligned}$$

Denote, as before, $s_{n-1,k} = \sum_{\varepsilon\in\mathcal{W}_{n-1,k}} \mathrm{m}(I_\varepsilon)$. It follows that

$$T_n = -\sum_{k\in\mathcal{S}} h_\ell(L_k)s_{n-1,k} + T_{n-1}.$$

Iterating, we get

$$\tau'(1) = \lim_{n\to+\infty} \sum_{k\in\mathcal{S}} h_\ell(L_k)\frac{1}{n}(s_{n-1,k} + s_{n-2,k} + \ldots + s_{1,k})$$

Observe now that $s_{n-1,k}$ is the $k^{\text{th}}$ component of $S_{n-1} = S_1P^{n-2}$ (see (3.4)). It is well known that the Cesaro means converge to the stationary distribution $\pi$ (even in the periodic case). The theorem follows. □

**Example 3.9.** Let

$$P = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

An analytic formula for $\tau(q)$ is complicated. Nevertheless, a numerical evaluation of $\tau(0)$ (which is also the box dimension of the support of m) is possible and Theorem 3.8 allows us to estimate the dimension of the measure m. We find

$$\dim \mathrm{m} \approx 0.58 \qquad \text{and} \qquad \dim_B(\operatorname{supp} \mathrm{m}) \approx 0.60.$$

This example shows that the "uniform" transition densities does not need to imply the maximality of $\dim \mathrm{m}$. Indeed, we will see in Corollary 4.2 that there always exists a choice of the transition matrix for which the dimension of the measure coincides with the dimension of its support.

## 4. Multifractal analysis and dimension of the support

By the definition of $\tau$ we have

$$\tau(0) = \limsup_{n\to\infty} \frac{\log N_n}{n \log \ell},$$

where $N_n$ is the number of intervals from $\mathcal{F}_n$ having positive measure. In our context the limit exists, so $\tau(0) = \dim_B(\operatorname{supp} \mathrm{m})$. Observe that the support of the measure m doesn't depend on the specific values of $p_{ij}$ but only on the configuration of the nonzero entries in the matrix $P_0$ and in the initial distribution $p = (p_0, \cdots, p_{\ell-1})$ . More precisely, the support of the measure m is the compact set

$$K = \bigcap_{n\geq 1} \bigcup_{p_{\varepsilon_1} p_{\varepsilon_1\varepsilon_2}\cdots p_{\varepsilon_{n-1}\varepsilon_n}>0} \overline{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

Indeed, the construction of the support can be viewed as a Cantor-like removal process. Given an interval $I_{\varepsilon_1\cdots\varepsilon_n}$ of the $n^{\text{th}}$ generation with $\varepsilon_n = i$, its $j^{\text{th}}$ subinterval will be removed if and only if $p_{ij} = 0$ (cf. Example 3.1 (1)).

According to Theorem 3.2, $\overline{\dim}_B(\operatorname{supp} \mathrm{m}) = \log_\ell \lambda_0$ where $\lambda_0$ is the spectral radius of the matrix $P_0$, so that the box dimension of $\operatorname{supp} \mathrm{m}$ does not depend on the initial distribution $p$ and only depends on the configuration of the nonzero entries of the matrix $P_0$.

This motivates the following questions. Given a configuration of nonzero entries of $P_0$ and of the initial distribution $p$, which values of $p_{ij}$ maximize $\dim \mathrm{m}$? Is the maximal value of $\dim \mathrm{m}$ equal to the box dimension of the support ? Is this maximal measure unique?

In some cases one has an immediate answer. In particular, Example 3.7 says that if each row of the matrix $P$ has the same number $\kappa$ of nonzero entries, the maximum of $\dim \mathrm{m}$ is obtained when each nonzero entry of the matrix $P$ is equal to $1/\kappa$ and is then equal to $\frac{\log \kappa}{\log \ell}$.

The general answer will be a consequence of the following result which says that the measure m satisfies the multifractal formalism.

**Theorem 4.1.** *Let $X$ be an ireductible Markov chain with transition matrix $P$ and let* m *be the associated measure. Then* m *satisfies the multifractal formalism. More precisely, define*

$$E_\alpha = \left\{ x \in [0,1] \ ; \ \lim_{n\to\infty} \frac{\log \mathrm{m}(I_n(x))}{\log |I_n(x)|} = \alpha \right\} \ .$$

*Then, for any $-\tau'(+\infty) < \alpha < -\tau'(\infty)$,*

$$\dim(E_\alpha) = \tau^*(\alpha),$$

*where* $\tau^*(\alpha) = \inf_q(\alpha q + \tau(q))$ *is the Legendre transform of the function* $\tau$.

*Proof.* We will prove the existence of a Gibbs measure at a given state $q$, that is an auxiliary measure $\mathrm{m}_q$ such that for any $\ell$-adic interval $I$ one has

$$\mathrm{m}_q(I) \asymp |I|^{\tau(q)}\,\mathrm{m}(I)^q. \tag{4.1}$$

Note that in (4.1), the constant may depend on $q$. Since the function $\tau$ is differentiable, it is well known that the existence of such a measure at each state $q$ implies the validity of the multifractal formalism for m (see for example [2] or [8]).

Now again, such a Gibbs measure will be obtained with an elementary construction. Note that $P$ is irreductible if and only if $P_q$ is. By Perron-Frobenius Theorem, the spectral radius $\lambda_q$ of $P_q$ is a simple eigenvalue and there exists a unique probability vector $\pi_q$ with strictly positive entries and satisfying $P_q\pi_q = \lambda_q\pi_q$.

Define $D_q$ as the $\ell \times \ell$-matrix having the coordinates of $\pi_q$ on the diagonal and zeros elsewhere. Set $Q_q = \dfrac{1}{\lambda_q} D_q^{-1} P_q D_q$ and let $\mathbf{1}$ be the column vector $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^\ell$. Then we have

$$Q_q\mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q D_q \mathbf{1} = \frac{1}{\lambda_q} D_q^{-1} P_q \pi_q = D_q^{-1}\pi_q = \mathbf{1}.$$

In other words, $Q_q$ is a stochastic matrix and thus it can be associated to a Markov chain $X^{(q)}$. We may and do assume that this chain has the initial distribution $q_i = \alpha p_i^q$, $i \in \mathcal{S}$, where $(p_0, \dots, p_{\ell-1})$ is the initial distribution of $X$ and $\alpha = (p_0^q + \cdots + p_{\ell-1}^q)^{-1}$ is the normalization constant. Let $\mathrm{m}_q$ be the measure induced by $X^{(q)}$. Remark that if $Q_q = (q_{ij})$, $D_q = (d_{ij})$, then we have

$$q_{ij} = \frac{1}{\lambda_q} d_{ii}^{-1} p_{ij}^q d_{jj}, \qquad i, j \in \mathcal{S}.$$

Cleraly, $q_{ij} > 0$ if and only if $p_{ij} > 0$ and $q_i > 0$ if and only if $p_i > 0$. Therefore the measures $\mathrm{m}_q$ and m have the same support.

Let $I = I_{\varepsilon_1\cdots\varepsilon_n} \in \mathcal{F}_n$. We have

$$\begin{aligned} m_q(I) &= q_{\varepsilon_1} q_{\varepsilon_1\varepsilon_2} \cdots q_{\varepsilon_{n-1}\varepsilon_n} \\ &= \alpha p_{\varepsilon_1}^q \left(\frac{1}{\lambda_q} d_{\varepsilon_1\varepsilon_1}^{-1} p_{\varepsilon_1\varepsilon_2}^q d_{\varepsilon_2\varepsilon_2}\right) \cdots \left(\frac{1}{\lambda_q} d_{\varepsilon_{n-1}\varepsilon_{n-1}}^{-1} p_{\varepsilon_{n-1}\varepsilon_n}^q d_{\varepsilon_n\varepsilon_n}\right) \\ &= \frac{\alpha}{\lambda_q^{n-1}} d_{\varepsilon_1\varepsilon_1}^{-1} m(I_{\varepsilon_1\cdots\varepsilon_n})^q d_{\varepsilon_n\varepsilon_n} \end{aligned}$$

Further, since the entries of the the eigenvector $\pi_q$ are strictly positive, there is a constant $c$ (possibly depending on $q$) such that

$$c^{-1} \le \frac{d_{ii}}{d_{jj}} \le c$$

for any $i, j \in \mathcal{S}$. This yields

$$m_q(I) \asymp \frac{1}{\lambda_q^{n-1}} m(I)^q.$$

Now, observe that by Theorem 3.2, $\ell^{\tau(q)} = \lambda_q$ so that

$$|I|^{\tau(q)} = \left(\ell^{-n}\right)^{\tau(q)} = \lambda_q^{-n}.$$

It follows that for any $I \in \mathcal{F}_n$ we have

$$m_q(I) \asymp |I|^{\tau(q)} m(I)^q.$$

Note that in the above estimate the implicit constants may depend on $q$ but not on $n$ and $I$. Therefore $m_q$ is the needed Gibbs measure and the theorem follows. $\square$

**Corollary 4.2.** *Let $K$ be the support of the measure $\mathrm{m}$. The measure $\overline{\mathrm{m}} = \mathrm{m}_0$ associated to the matrix $Q_0$ satisfies $\operatorname{supp} \overline{\mathrm{m}} = K$, is monofractal and strongly equivalent to the Hausdorff measure $\mathcal{H}^{\tau(0)}$ on $K$. In particular,*

$$\dim(\overline{\mathrm{m}}) = \dim_H(K) = \dim_B(K)$$

*and $\overline{\mathrm{m}}$ is a measure driven by a Markov chain, with support $K$ and with maximal dimension.*

*Proof.* Let $\overline{\mathrm{m}} = \mathrm{m}_0$ from the previous proof. Then we have

$$\overline{\mathrm{m}}(I) \asymp |I|^{\tau(0)}$$

for any $\ell$-adic interval $I$ such that $\mathrm{m}(I) > 0$. In particular, for any $x \in K$, $\overline{\mathrm{m}}\,(I_n(x)) \asymp |I_n(x)|^{\tau(0)}$. By Billingsley's theorem (see e.g. [4], Propositions 2.2 and 2.3), we conclude that $\overline{\mathrm{m}}$ is equivalent to $\tau(0)$-dimensional Hausdorff measure $\mathcal{H}^{\tau(0)}$ on $K$. In particular $\mathcal{H}^{\tau(0)}(K)$ is positive and finite. It follows that

$$\dim_H(K) = \tau(0) = \dim_B(K).$$

On the other hand, the structure function $\overline{\tau}$ of the measure $\overline{\mathrm{m}}$ is $\overline{\tau}(q) = \tau(0)(1-q)$. It follows that the measure $\overline{\mathrm{m}}$ is monofractal, and that

$$\dim \overline{\mathrm{m}} = -\overline{\tau}'(1) = \tau(0) = \dim_B(K).$$

$\square$

**Example 4.3.** Suppose that $X$ is a random walk on $\mathbb{Z}_\ell$ (cf. Example 3.1 (3)). Then $\tau(q) = (1-q)\log_\ell 2$ and

$$\dim \mathrm{m} = -\tau'(1) = \log_\ell 2 = \tau(0) = \dim_B(K).$$

**Example 4.4.** Let

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

It can be easily seen that $P$ is irreducible (actually, $P^2$ has only positive entries). We obtain

$$\tau(q) = -\log_3 2 + \log_3 \left(2^{-q} + 3^{-q} + \sqrt{4^{-q} + 6^{1-q} + 9^{-q}}\right).$$

Hence

$$\dim_H(\operatorname{supp} \mathrm{m}) = \log_3(1+\sqrt{2}) \approx 0.802 \quad \text{and} \quad \dim \mathrm{m} = \frac{1}{7}(3 + 4\log_3 2) \approx 0.789.$$

As observed in Example 3.9, the transitions are uniform row by row, but the measure m is not monofractal. The Markov chain inducing the maximal measure $\overline{\mathrm{m}}$ is associated to the following transition matrix :

$$Q_0 = (\sqrt{2}-1) \times \begin{pmatrix} 1 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 \end{pmatrix}.$$

## 5. Invariance, ergodicity and application to the uniqueness of $\overline{\mathrm{m}}$

The goal of this section is to discuss the uniqueness of measure $\overline{\mathrm{m}}$ with maximal dimension given in Corollary 4.2. This is the object of Theorem 5.5.

We need to start with some preliminary results. Let us introduce the shift $\sigma$ on $[0,1)$ defined by $\sigma(x) = \ell x - E(\ell x)$, where $E(y)$ is the integer part of $y$. Observe that $\sigma\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = I_{\varepsilon_2\cdots\varepsilon_n}$ and that

$$\sigma(x) = \bigcap_n I_{\varepsilon_2\cdots\varepsilon_n} \quad \text{if} \quad x = \bigcap_n I_{\varepsilon_1\cdots\varepsilon_n}.$$

That is why $\sigma$ is called the shift.

**Proposition 5.1.** *Let $P$ be an irreducible $\ell\times\ell$ transition matrix, $\nu$ be the (unique) probability vector such that $\nu P = \nu$ and $X$ be the Markov chain with transition $P$ and initial law $\nu$. Set $\mathrm{m}_P$ to be the probability measure driven by the Markov chain $X$. Then, $\mathrm{m}_P$ is $\sigma$-invariant and ergodic.*

*Proof.*

$$\begin{aligned}
\mathrm{m}_P(\sigma^{-1}(I_{\varepsilon_1\cdots\varepsilon_n})) &= \sum_{j=0}^{\ell} \mathrm{m}_P(I_{j\varepsilon_1\cdots\varepsilon_n}) \\
&= \sum_{j=0}^{\ell} \nu_j p_{j\varepsilon_1}\cdots p_{\varepsilon_{n-1}\varepsilon_n} \\
&= \nu_{\varepsilon_1} p_{\varepsilon_1\varepsilon_2}\cdots p_{\varepsilon_{n-1}\varepsilon_n} \\
&= \mathrm{m}_P(I_{\varepsilon_1\cdots\varepsilon_n})
\end{aligned}$$

So, by the monotone class theorem, the measure $\mathrm{m}_P$ is $\sigma$-invariant.

Let $k$ be an integer such that $P + P^2 + \cdots + P^k$ has strictly positive entries. We claim that there exists a constant $C > 0$ such that for any $I, J \in \bigcup_n \mathcal{F}_n$, we have

$$\frac{1}{C}\,\mathrm{m}_P(I)\,\mathrm{m}_P(J) \le \sum_{j=0}^{k-1}\sum_{K\in\mathcal{F}_j} \mathrm{m}_P(IKJ) \le C\,\mathrm{m}_P(I)\,\mathrm{m}_P(J). \tag{5.1}$$

Indeed, if $I = I_{\varepsilon_1\cdots\varepsilon_n}$, $J = I_{\delta_1\cdots\delta_m}$, and if $(\pi_{ij})_{i,j}$ denote the coefficients of the matrix $P + P^2 + \cdots + P^k$, it is easy to check that

$$\sum_{j=0}^{k-1}\sum_{K\in\mathcal{F}_j} \mathrm{m}_P(IKJ) = \nu_{\varepsilon_1} p_{\varepsilon_1\varepsilon_2}\cdots p_{\varepsilon_{n-1}\varepsilon_n} \times \pi_{\varepsilon_n\delta_1} p_{\delta_1\delta_2}\cdots p_{\delta_{m-1}\delta_m}$$

and the claim follows.

Inequality (5.1) can be rewritten as

$$\sum_{j=0}^{k-1} \mathrm{m}_P\left(I \cap \sigma^{-(n+j)}(J)\right) \asymp \mathrm{m}_P(I) \times \mathrm{m}_P(J) \tag{5.2}$$

where $n$ is the generation of $I$. If we observe that any open set is a countable union of disjoint intervals in $\cup_n \mathcal{F}_n$, inequality (5.2) remains true when $J$ is an open set.

Finaly, by regularity of the measure $\mathrm{m}_p$, it is also true for any Borel set $J$. In particular, if $E$ is a $\sigma$-invariant Borel set, we get

$$\forall I \in \bigcup_n \mathcal{F}_n, \qquad k\,\mathrm{m}_P(I\cap E) \asymp \mathrm{m}_P(I)\times \mathrm{m}_P(E).$$

Again, it remains true when $I$ is an arbitrary Borel set. In particular,

$$\mathrm{m}_P\left(([0,1]\setminus E)\cap E\right) \asymp \mathrm{m}_P\left([0,1]\setminus E\right)\times \mathrm{m}_P(E)$$

which proves that $\mathrm{m}_P(E)=0$ or $\mathrm{m}_P\left([0,1]\setminus E\right)=0$. □

*Remark* 5.2. Inequality 5.2 is a particular case of the so called weak quasi-Bernoulli property which was introduced by B. Testud in [11].

**Corollary 5.3.** *Let $P$ and $\tilde{P}$ be two different irreducible $\ell\times\ell$ transition matrices. Then $\mathrm{m}_P$ is singular with respect to $\mathrm{m}_{\tilde{P}}$.*

*Proof.* According to the ergodic theorem, it suffices to show that $\mathrm{m}_P\neq\mathrm{m}_{\tilde{P}}$. Let $\nu$ and $\tilde{\nu}$ be the invariant distributions of the stochastic matrix $P$ and $\tilde{P}$. If $\nu_i\neq\tilde{\nu}_i$ for some $i$, then $\mathrm{m}_P(I_i)\neq\mathrm{m}_{\tilde{P}}(I_i)$. If $\nu=\tilde{\nu}$ and if $p_{ij}\neq\tilde{p}_{ij}$, we can write :

$$\mathrm{m}_P(I_{ij})=\nu_i p_{ij}\neq\tilde{\nu}_i\tilde{p}_{ij}=\mathrm{m}_{\tilde{P}}(I_{ij}).$$

□

**Corollary 5.4.** *Let $P$, $\tilde{P}$ be two irreducible $\ell\times\ell$ transition matrices, $\mathrm{p}$ and $\tilde{\mathrm{p}}$ to be two probability vectors. Set $\mathrm{m}$ and $\tilde{\mathrm{m}}$ be the associated measures. Suppose that* $\mathrm{supp}(\mathrm{m})=\mathrm{supp}(\tilde{\mathrm{m}})$. *Then, there are only two possible cases :*

(1) *$P=\tilde{P}$ and the measures $\mathrm{m}$ and $\tilde{\mathrm{m}}$ are strongly equivalent (i.e. $\mathrm{m}\asymp\tilde{\mathrm{m}}$)*
(2) *$P\neq\tilde{P}$ and the measures $\mathrm{m}$ and $\tilde{\mathrm{m}}$ are mutually singular.*

*Proof.* Let $\mathcal{A}\subset\mathcal{S}$ be the set of ranges of the nonzero entries of $p$ (which is also the set of ranges of nonzero entries of $\tilde{p}$). Let $F=\bigcup_{\varepsilon\in\mathcal{A}}\overline{I_\varepsilon}$. We claim that $\mathrm{m}$ is strongly equivalent to the measure $\mathrm{m}_P$ restricted to $F$. This is an easy consequence of the fact that the invariant probability vector $\nu$ (satisfying $\nu P=\nu$) has strictly positive entries. In the same way, $\tilde{\mathrm{m}}$ is strongly equivalent to the measure $\mathrm{m}_{\tilde{P}}$ restricted to $F$. Corollary 5.4 is then a consequence of Corollary 5.3. □

Now, we are able to prove the following theorem on the measure $\overline{\mathrm{m}}$ given by Corollary 4.2.

**Theorem 5.5.** *Let $P_0=\left(p^0_{ij}\right)$ be an irreductible $\ell\times\ell$ matrix such that $p^0_{ij}\in\{0,1\}$ for any $ij$ and let $\mathrm{p}^0=\left(\mathrm{p}^0_0,\cdots,\mathrm{p}^0_{\ell-1}\right)$ be a line vector such that $\mathrm{p}^0_i\in\{0,1\}$ for any $i$. Suppose that $\mathrm{p}^0\neq(0,\cdots,0)$ and define the compact set $K$ by*

$$K=\bigcap_{n\geq 1}\ \bigcup_{\mathrm{p}^0_{\varepsilon_1}p^0_{\varepsilon_1\varepsilon_2}\cdots p^0_{\varepsilon_{n-1}\varepsilon_n}=1}\overline{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

*Let $\delta=\dim_H(K)$ and let $\mathrm{m}$ be a measure with support $K$, driven by a Markov chain $X$ with irreductible transition matrix $P$. Then, $\dim\mathrm{m}=\delta$ if and only if*

$$P=\frac{1}{\lambda_0}D_0^{-1}P_0D_0$$

*where $\lambda_0$ is the spectral radius of $P_0$ and $D_0$ is the diagonal matrix whose diagonal entries are the coordinates of the (unique) probability vector $\pi_0$ satisfying $P_0\pi_0=$*

$\lambda_0\pi_0$. *Moreover, the case* $P = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$ *is the only case where the measure* m *is monofractal.*

*Proof.* Assume for simplicity that $p_i^0 = 1$ for any $i$. The general case is a standard modification. Remember that the support of the measure m only depends on the positions of the non-zero entries of the matrix $P$. It follows the non-zero entries of the matrices $P$ and $P_0$ hare located at the same places. Let $Q_0 = \frac{1}{\lambda_0} D_0^{-1} P_0 D_0$.

Suppose that $P = Q_0$. According to Corollary 4.2 and Corollary 5.4, the measure m is strongly equivalent to $\overline{\mathrm{m}} = \mathrm{m}_0$ which satisfies $\overline{\mathrm{m}}(I_n(x)) \asymp |I_n(x)|^\delta$ for any $x \in K$. In particular, m is such that $\dim \mathrm{m} = \delta$ and is monofractal.

Suppose now that $P \neq Q_0$. Corollary 5.4 says that m is singular with respect to $\mathrm{m}_0$ and we have to prove that $\dim \mathrm{m} < \delta$. Let

$$\tau(q) = \lim_{n\to+\infty} \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} \mathrm{m}(I)^q \right).$$

Recall that $\tau$ is analytic and such that $\tau(0) = \delta$ and $\tau(1) = 0$. Using the convexity of $\tau$, it is clear that

$$\tau'(1) = -\delta \iff \forall q \in [0,1],\ \tau(q) = \delta(1-q) \iff \forall q \in \mathbb{R},\ \tau(q) = \delta(1-q).$$

In order to prove that $\dim \mathrm{m} < \delta$, it is then sufficient to establish that $\tau(2) > -\delta$.

Denote by $I_0, \cdots, I_{\ell-1}$ the intervals of the first generation $\mathcal{F}_1$. If $j \in \{0, \cdots, \ell-1\}$ and $n \geq 1$, let $\mathcal{F}_n(j)$ be the intervals of $\mathcal{F}_n$ that are included in $I_j$. The measures m and $\mathrm{m}_0$ being mutually singular, we know that for dm-almost every $x \in K$,

$$\lim_{n\to+\infty} \frac{\mathrm{m}_0(I_n(x))}{\mathrm{m}(I_n(x))} = 0,$$

which can be rewritten as

$$\lim_{n\to+\infty} \frac{\ell^{-n\delta}}{\mathrm{m}(I_n(x))} = 0.$$

Using Egoroff's theorem in each $I_j$, we can find a set $A \subset K$ such that $\mathrm{m}(A \cap I_j) \geq \frac{1}{2}\mathrm{m}(I_j)$ for any $j \in \{0, \cdots, \ell-1\}$ and satisfying

$$\forall \varepsilon > 0, \quad \exists n_0 \geq 1; \quad \forall n \geq n_0,\ \forall x \in A, \quad \mathrm{m}(I_n(x)) \geq \frac{1}{\varepsilon}\ell^{-n\delta}.$$

It follows that

$$\sum_{J \in \mathcal{F}_{n_0}(j)} \mathrm{m}(J)^2 \geq \sum_{J \in \mathcal{F}_{n_0}(j)\ ;\ J\cap A \neq \emptyset} \frac{1}{\varepsilon}\ell^{-n_0\delta}\,\mathrm{m}(J) \geq \frac{1}{2\varepsilon}\ell^{-n_0\delta}\,\mathrm{m}(I_j).$$

Now, let $I \in \mathcal{F}_k$ and suppose that $I_{\varepsilon_1,\cdots,\varepsilon_k}$ with $\varepsilon_k = i$. Observe that if $J \in \mathcal{F}_n(j)$, then $\mathrm{m}(IJ) = \frac{p_{ij}}{\mathrm{m}(I_j)}\mathrm{m}(I)\,\mathrm{m}(J)$. If we choose $j_i \in \{0, \cdots, \ell-1\}$ such that $p_{ij_i} \neq 0$, we get

$$\begin{aligned}
\sum_{J \in \mathcal{F}_{n_0}} \mathrm{m}(IJ)^2 &\geq \sum_{J \in \mathcal{F}_{n_0}(j_i)} \mathrm{m}(IJ)^2 \\
&\geq \frac{p_{ij_i}^2}{\mathrm{m}(I_{j_i})^2}\mathrm{m}(I)^2 \sum_{J \in \mathcal{F}_{n_0}(j_i)} \mathrm{m}(J)^2 \\
&\geq \frac{p_{ij_i}^2}{2\varepsilon\,\mathrm{m}(I_{j_i})}\ell^{-n_0\delta}\,\mathrm{m}(I)^2.
\end{aligned}$$

Let $\varepsilon = \inf_i \left( \frac{p_{ij_i}^2}{4\,\mathrm{m}(I_{j_i})} \right)$ and the corresponding $n_0$. If $\eta$ is such that $\ell^{n_0\eta} = 2$, we can rewrite the last inequality as

$$\sum_{J\in\mathcal{F}_{n_0}} \mathrm{m}(IJ)^2 \geq \ell^{-n_0(\delta-\eta)}\,\mathrm{m}(I)^2.$$

If we sum this inequality on every interval $I$ of the same generation and iterate the process, we get for any $p \geq 1$

$$\sum_{I\in\mathcal{F}_{pn_0}} \mathrm{m}(I)^2 \geq \ell^{-(p-1)n_0(\delta-\eta)} \sum_{I\in\mathcal{F}_{n_0}} \mathrm{m}(I)^2 = C\ell^{-pn_0(\delta-\eta)}$$

which gives

$$\tau(2) \geq -(\delta-\eta) > -\delta.$$

Moreover, it is clear that $\tau(\mathbb{R})$ is not reduced to a single point. If follows that the measure m is multifractal. □

## References


1. F. Ben Nasr, I. Bhouri, Y. Heurteaux, *The validity of multifractal formalism: results and examples*, Adv. Math., **165**(2002), 264–284.
2. G. Brown, G. Michon and J. Peyrière. *On the Multifractal Analysis of Measures*, J. Stat. Phys., **66**(1992), 775–790.
3. R. Cawley and R.D. Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math., **92**(1992), 196–236.
4. K. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, 1997.
5. D. J. Feng and K. S. Lau, *The pressure function for products of nonnegative matrices*, Math. Res. Lett., **9**(2002), 363–378.
6. D.J. Feng and E. Olivier, *Multifractal analysis of weak Gibbs measures and phase transition-application to some Bernoulli convolution*, Ergodic Theory Dynam. Systems, **23** (2003), 1751–1784.
7. U. Frisch and G. Parisi, *On the singularity structure of fully developed turbulence*, Turbulence and predictability in geophysical fluid dynamics, Proceedings of the International Summer School in Physics Enrico Fermi, North Holland (1985), 84–88.
8. Y. Heurteaux, *Dimension of measures : the probabilistic approach*, Publ. Mat. **51** (2007), 243–290.
9. L. Olsen, *A multifractal formalism*, Adv. Math., **116** (1995), 82–196.
10. M. Reed and B. Simon, *Lectures on modern mathematical physics*, Academic Press, 1980.
11. B. Testud, *Mesures quasi-Bernoulli au sens faible : résultats et exemples*, Ann. Inst. H. Poincaré Probab. Statist. **42**(2006), 1–35.
12. C. Tricot, *Géométries et mesures fractales : une introduction*, Ellipses, 2008.
13. P. Tukia, *Hausdorff dimension and quasisymetric mappings*, Math. Scand., **65** (1989), 152–160.



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